The ordinates A_0, A_1, ..., A_{10} may be easily computed by means of a table of Hyperbolic Functions.

In this solution it has been assumed that the unwinding of the string begins at the vertex of the cone. If the unwinding begins at the base of the cone, we replace, in (3), θ by $(2\pi n-\theta)$ and take the negative sign of the radical since it is then a decreasing function of θ. This gives

$$s=\frac{R}{4\pi n}\Big[2\pi n\surd(4\pi^2n^2+k^2)-(2\pi n-\theta)\surd[(2\pi n-\theta)^2+k^2]$$

$$+k^2\log\Big(\frac{2\pi n+\surd(4\pi^2n^2+k^2)}{2\pi n-\theta+\surd[(2\pi n-\theta)^2+k^2]}\Big)\Big].$$

We then have, $ds_1=[(s+\dfrac{(2\pi n-\theta)Rd\theta}{2\pi n})^2+ds^2+\dfrac{l^2-R^2}{4\pi^2n^2}d\theta^2]^{\frac{1}{2}}$ ....(7).

But $ds=-\dfrac{R}{2\pi n}\surd[(2\pi n-\theta)^2+k^2]d\theta$.

Substituting the values of s and ds in (7) and letting $LR/4\pi n$=the entire length of the string, and $\theta=2\pi n-k\sinh[(1-x)\log a]$, where

$$a=\frac{2\pi n+\surd(4\pi^2n^2+k^2)}{k},$$

we have,

$$s_1=\frac{R}{4\pi n}\int_0^1\Big[\{L-\tfrac{1}{2}k^2\sinh[2(1-x)\log a]+k^2(1-x)\log a$$

$$+2k\sinh[(1-x]\log a]\}^2+4k^2\sinh^2[(1-x)\log a]$$

$$+8k^2-4\Big]^{\frac{1}{2}}\cosh[(1-x)\log a]dx,$$

the value of which may be obtained by the foregoing method of approximation.

DIOPHANTINE ANALYSIS.

126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

Remark by DR. L. E. DICKSON, The University of Chicago.

According to A. H. Holmes (MONTHLY, 1905, p. 141), the program could be arranged in 7 evenings, A and B being partners only the first evening. But this solution is clearly erroneous, since by the last condition of the problem A and B shall play against the 15 possible pairs of the remaining six.

I proceed to prove that it is impossible to construct a program of the desired kind. The notation may be chosen so that the order of play for the first evening is

(I) 12, 34, 56, 78.

With 12, 35, must go either 47, 68 or else 48, 67 (since 46, 78 is excluded by I). But these two cases are interchanged by the substitution (78), which does not alter I. Hence, if the problem is possible, there would be a program with I and

(II) 12, 35, 47, 68.

Then with 12, 36 cannot go 47, 58 or 45, 78. In this way we get

(III) 12, 36, 48, 57;
(IV) 12, 37, 46, 58;
(V) 12, 38, 45, 67.

With 13, 24 goes 57, 68 or 58, 67, cases interchanged by the substitution (5768), which leaves I unaltered and permutes II, IV, III, V, cyclically. Hence we may set

(VI) 13, 24, 57, 68;
(VII) 14, 23, 58, 67 (by I and VI);
(VIII) 16, 23, 45, 78 (by III and VII).

Then 15, 23 cannot go with 46, 78; 47, 68; or 48, 67, by VIII, II, VII, respectively. Hence, the problem is impossible.

First modification of problem. If we allow each couple to play exactly three times against every other couple, the problem becomes possible, there being

$$ {}_8C_2 \cdot {}_6C_2 \cdot {}_4C_2 \cdot {}_2C_2 \div 4! = 105 $$

orders of the players, corresponding to the 105 substitutions of the type (12)(34)(56)(78). While this program is absolutely fair to each player, it would require 105 evenings.

Second modification. Required a program for 35 evenings of duplicate whist between eight players, such that during the series every couple shall play every other couple once and but once, while in each evening there shall be three orders of play in which no two persons play together twice.

Such a program (which is equally fair to all players) is the following:

12, 36; 58, 47	13, 28; 45, 67	12, 35; 46, 78	13, 26; 48, 57
14, 38; 25, 67	15, 27; 38, 46	14, 26; 37, 58	14, 27; 35, 68
15, 26; 34, 78	17, 25; 36, 48	18, 27; 34, 56	16, 23; 45, 78
13, 25; 47, 68	15, 24; 68, 37	12, 34; 57, 68	12, 37; 48, 56
15, 23; 48, 67	16, 28; 35, 47	13, 24; 56, 78	14, 28; 36, 57
16, 24; 38, 57	18, 23; 57, 46	17, 26; 38, 45	18, 24; 35, 67

with cyclic permutations of the last three columns;

12, 67; 35, 48	12, 45; 36, 78	12, 38; 47, 56	13, 27; 45, 68
13, 46; 25, 78	14, 56; 28, 37	17, 23; 46, 58	14, 23; 58, 67
15, 47; 23, 68	17, 68; 25, 34	18, 45; 27, 36	17, 56; 28, 34
13, 58; 26, 47	14, 78; 26, 35	15, 36; 27, 48	16, 27; 34, 58
15, 28; 34, 67	16, 25; 38, 47	16, 37; 24, 58	17, 28; 36, 45
18, 37; 25, 46	17, 58; 24, 36	17, 35; 28, 46	18, 47: 23, 56
16, 34; 27, 58	16, 58; 23, 47	16, 48; 25, 37	
17, 24; 38, 56	17, 34; 28, 56	17, 46; 23, 58	
18, 26; 37, 45	18, 25; 37, 46	18, 36; 27, 45	

This solution, which I made in 1905, has been re-checked at two different times. Note that the first 24 orders furnish a Thompson program for 24 evenings, each couple to play against the other three couples in turn. It would be interesting to know whether or not 24 is a maximum* in the Thompson problem.

134. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

How many sets of solutions has the congruence $\lambda+\mu+\nu+\xi\equiv 0 \pmod{p-1}$ p being a prime number; the order of λ, μ, ν, ξ being disregarded.

Solution by the PROPOSER.

Assume that $p>5$, and let n_i be the number of solutions in which i of the λ, μ, ν, ξ are congruent to each other (mod $p-1$). If $p\equiv 1$ (mod 4), then $n_4=4$, viz.,

$$\lambda\equiv\mu\equiv\nu\equiv\xi\equiv 0,\ \frac{p-1}{4},\ \frac{p-1}{2},\ \frac{3(p-1)}{4}.$$

If $p\equiv 3$ (mod 4), $n_4=2$, viz., $\lambda\equiv\mu\equiv\nu\equiv\xi\equiv 0,\ \dfrac{p-1}{2}$.

Next, let $i=3$, so that the congruence reduces to $\lambda+3\nu\equiv 0 \pmod{p-1}$.

*Note that a Thompson program for 22 evenings is given by (I)—(VIII) and

13, 25, 46, 78;	13, 26, 47, 58;	13, 27, 48, 56;	14, 25, 37, 68;
14, 26, 38, 57;	15, 24, 36, 78;	15, 27, 34, 68;	16, 28, 34, 57;
17, 24, 38, 56;	17, 26, 35, 48;	17, 28, 36, 45;	18, 23, 46, 57;
18, 24, 35, 67;	18, 25, 36, 47;	18, 26, 37, 45.	

These 22 orders are mutually consistent, while no other is consistent with them.